ECE 307 – Techniques for Engineering Decisions

Lecture 5. Networks and Flows

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NETWORKS AND FLOWS

- A network is a system of lines or channels or branches that connect different points
- Examples abound in nearly all aspects of life:
  - electrical systems;
  - communication networks;
  - airline webs;
  - local area networks; and
  - distribution systems

NETWORKS AND FLOWS

- **The network structure is also common to many other systems that at first glance are not necessarily viewed as networks**
  - **distribution of products through a system consisting of manufacturing plants, warehouses and retail outlets**
  - **matching problems such as work to people, tasks to machines and computer dating**

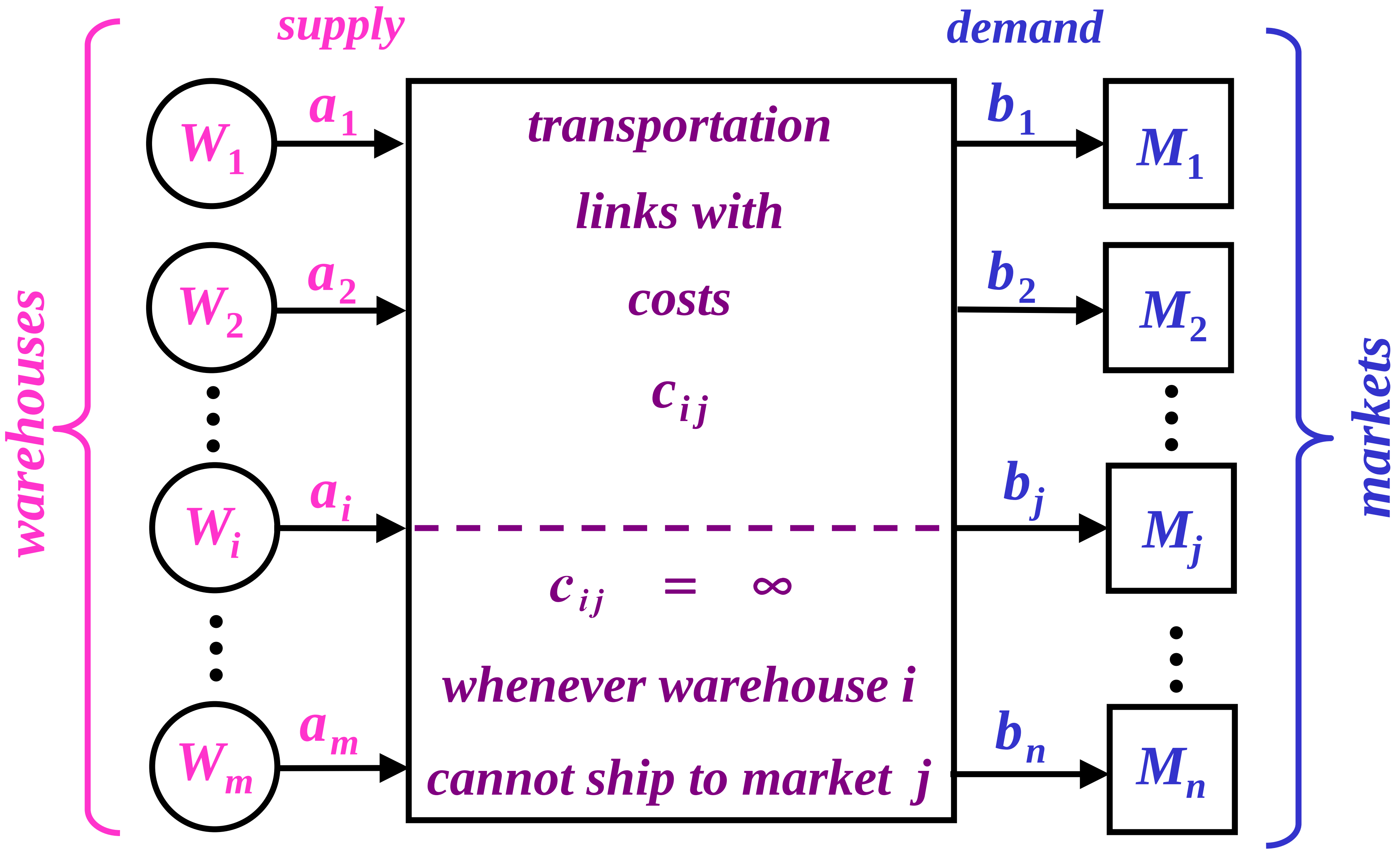
NETWORKS AND FLOWS

  - river systems with pondage for electricity generation
  - mail delivery networks
  - freight delivery networks
  - project management of multiple tasks in a large undertaking such as a major construction project or a space flight
- We consider a broad range of network and network flow problems

THE TRANSPORTATION PROBLEM

- ☐ **The basic idea of the transportation problem is illustrated with the problem of the distribution of a specified *homogeneous* product from several warehouses to a number of localities *at least cost***
- ☐ **We consider a system with m warehouses, n markets and links between them with the specified costs of transportation**

THE TRANSPORTATION PROBLEM



supply

demand

warehouses

W_1 a_1 b_1 M_1

W_2 a_2 b_2 M_2

W_i a_i b_j M_j

W_m a_m b_n M_n

markets

transportation links with costs

c_{ij}

$c_{ij} = \infty$

whenever warehouse i cannot ship to market j

THE TRANSPORTATION PROBLEM

- all the supply comes from the m warehouses; we associate the index $i = 1, 2, \ldots , m$ with a warehouse
- all the demand is at the n markets; we associate the index $j = 1, 2, \ldots , n$ with a market
- shipping costs c_{ij} for each unit from the warehouse i to the market j

THE TRANSPORTATION PROBLEM

- ☐ **The transportation problem is to determine the *optimal shipping schedule* that minimizes shipping costs from the set of m warehouses to the set of n markets by determining the quantities shipped from each warehouse i to each market j , $i = 1, 2, \ldots , m$, $j = 1, 2, \ldots , n$**

LP FORMULATION OF THE TRANSPORTATION PROBLEM

- ☐ **The decision variables are defined to be**

$$x_{ij} = \textit{quantity shipped from warehouse } i \textit{ to market } j,$$

$$i = 1, 2, \ldots, m , \quad j = 1, 2, \ldots, n$$

- ☐ **The objective function is**

$$\min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

LP FORMULATION OF THE TRANSPORTATION PROBLEM

- **The constraints are:**

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \qquad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n$$

LP FORMULATION OF THE TRANSPORTATION PROBLEM

- ☐ **Note that feasibility requires that**

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$$

- ☐ **When**

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

all available supply at the m warehouses is shipped to meet all the demands of the n markets; this is known as the *standard transportation problem*

STANDARD TRANSPORTATION PROBLEM (STP)

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

$s.t.$

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i \\ \sum_{i=1}^{m} x_{ij} &= b_j \\ x_{ij} &\geq 0 \end{aligned}\right\} \qquad \begin{aligned} i &= 1, \dots, m \\ j &= 1, \dots, n \end{aligned}$$

STANDARD TRANSPORTATION PROBLEM (STP)

- ☐ The standard transportation problem has
  - ○ mn variables x_{ij}
  - ○ $m + n$ equality constraints
- ☐ However, since

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

there are at most $(m + n - 1)$ independent constraints and consequently at most $(m + n - 1)$ independent variables x_{ij} (*basic variables*)

TRANSPORTATION PROBLEM EXAMPLE

market *j* w/h *i*	M_1	M_2	M_3	M_4	supplies
W_1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	x_{14} c_{14}	a_1
W_2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	x_{24} c_{24}	a_2
W_3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	x_{34} c_{34}	a_3
demands	b_1	b_2	b_3	b_4	$\sum_i a_i = \sum_j b_j$

TRANSPORTATION PROBLEM NUMERICAL EXAMPLE

market j *w/h i*	M_1	M_2	M_3	M_4	a_I
W_1	2	2	2	1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
b_j	4	3	4	4	

THE LEAST – COST RULE PROCEDURE

- ☐ **This procedure generates an initial *basic feasible solution* which has at most $(m + n - 1)$ positive-valued *basic variables***
- ☐ **The principal idea of the scheme is to select, at each step, the variable x_{ij} with the *lowest shipping costs* c_{ij} as the next *basic variable* to enter the basis**

APPLICATION OF THE LEAST – COST RULE

- ☐ **c_{14} is the lowest c_{ij} and we select x_{14} as a *basic variable***
- ☐ **We choose x_{14} as large as possible without violating any constraints:**

$$min \ \{ a_1 , b_4 \} = min \ \{ 3 , 4 \} = 3$$

- ☐ **We set $x_{14} = 3$ and**

$$x_{11} = x_{12} = x_{13} = 0$$

- ☐ **We delete row 1 from any further consideration since all the supplies from W_1 are exhausted**

APPLICATION OF THE LEAST – COST RULE

market j / *w/h i*	M_1	M_2	M_3	M_4	a_i
W_1	2	2	2	③ 1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
b_j	4	3	4	4	

APPLICATION OF THE LEAST – COST RULE

- ☐ **The remaining demand at M_4 is**

$$4 - 3 = 1$$

which is the value for the modified demand at M_4

- ☐ **We again apply the *criterion selection* for the reduced tableau: since c_{24} is the lowest-valued c_{ij} , we select x_{24} as the next *basic variable***

APPLICATION OF THE LEAST – COST RULE

- ☐ **We wish to set x_{24} as large as possible without violating any constraints:**

$$min \; \{ a_2 , b_4 \} = min \; \{ 7 , 1 \} = 1$$

and we set $x_{24} = 1$ and since there is no more demand at M_4

$$x_{34} = 0$$

- ☐ **We delete column 4 from any further consideration since all the demand at M_4 is met**

APPLICATION OF THE LEAST – COST RULE

- ☐ **The remaining supply at W_2 is**

  $$7 - 1 = 6 ,$$

  which is the value for the modified supply at W_2

- ☐ **We repeat these steps until we find the values of the $(m + n - 1)$ nonzero *basic variables* to obtain a *basic feasible solution***

- ☐ **In the reduced tableau,**

APPLICATION OF THE LEAST – COST RULE

market j *w/h i*	M_1	M_2	M_3	a_i
W_2	10	8	④ 5	6
W_3	7	6	*0* 6	5
b_j	4	3	4	

APPLICATION OF THE LEAST – COST RULE

- **pick x_{23} to enter the basis as the next basic variable**
- **set**

  $$x_{23} = min \{ 6, 4 \} = 4$$

  and set $x_{33} = 0$
- **eliminate column 3 and reduce the supply at W_2 to**

  $$6 - 4 = 2$$

- **For the reduced tableau**

APPLICATION OF THE LEAST – COST RULE

market j w/h i	M_1	M_2	a_i
W_2	$\boxed{10}$	0 $\boxed{8}$	2
W_3	$\boxed{7}$	③ $\boxed{6}$	5
b_j	4	3	

APPLICATION OF THE LEAST – COST RULE

- pick x_{32} to enter the basis
- set

$$x_{32} = min\,\{\,3, 5\,\} = 3$$

and set $x_{22} = 0$

- eliminate column 2 in the reduced tableau and reduce the supply at W_3 to

$$5 - 3 = 2$$

- The last reduced tableau is

APPLICATION OF THE LEAST – COST RULE

market j w/h i	M_1	a_i
W_2	10	2
W_3	② 7	2
b_j	4	

APPLICATION OF THE LEAST – COST RULE

- pick x_{31} to enter the basis
- set

$$x_{31} = \mathit{min}\,\{\,2, 4\,\} = 2$$

- reduce the demand at M_1 to

$$4 - 2 = 2$$

- the value of

$$x_{21} = 2$$

is obtained by default

INITIAL *BASIC FEASIBLE SOLUTION*

market j *w/h i*	M_1	M_2	M_3	M_4	a_i
W_1	2	2	2	3 1	3
W_2	2 10	8	4 5	1 4	7
W_3	2 7	3 6	6	8	5
b_j	4	3	4	4	

APPLICATION OF THE LEAST – COST RULE

- ❑ The feasible solution involves only the basic variables and results in shipment costs of

$$\sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} = 1 \cdot 3 + 4 \cdot 1 + 5 \cdot 4 + 6 \cdot 3 + 7 \cdot 2 + 10 \cdot 2$$

$$= 79$$

THE *STP*

- The primal problem is

$$
\left.
\begin{array}{llll}
\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} & & & \\
s.t. & & & \\
u_i \leftrightarrow & \sum_{j=1}^{n} x_{ij} = a_i & i = 1, \ldots ,m & \\
v_j \leftrightarrow & \sum_{i=1}^{m} x_{ij} = b_j & j = 1, \ldots ,n & \\
 & x_{ij} \geq 0 & &
\end{array}
\right\} (P)
$$

THE *STP*

- **The dual problem is**

$$\left.\begin{array}{llll} \max W = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j & & & \\ s.t. & & & \\ x_{ij} & \leftrightarrow & u_i + v_j \leq c_{ij} & i = 1, \ldots, m \\ & & & j = 1, \ldots, n \\ & & u_i, v_j \text{ are unrestricted in sign} & \end{array}\right\} \quad (D)$$

THE *STP*

- ☐ The *complementary slackness conditions* for (D) are

$$x^{*}_{ij}[u^{*}_{i} + v^{*}_{j} - c_{ij}] = 0 \qquad \begin{matrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{matrix}$$

- ☐ Due to the equalities in (P), the *complementary slackness conditions* in (P) cannot provide any useful information

THE TRANSPORTATION PROBLEM

- ☐ The *complementary slackness conditions* obtain

$$x_{ij}^{*} > 0 \Rightarrow u_{i}^{*} + v_{j}^{*} = c_{ij}$$

$$u_{i}^{*} + v_{j}^{*} < c_{ij} \Rightarrow x_{ij}^{*} = 0$$

- ☐ We make use of these *complementary slackness conditions* to develop the so-called $u - v$ *method* for solving the *standard transportation problem*

THE $u - v$ METHOD

☐ The $u - v$ *method* starts with a *basic feasible solution* for the primal problem, determines the corresponding dual variables (as if the *basic feasible solution* were optimal) and uses the duals to determine the *adjacent basic feasible solution*; the process continues until the optimality conditions are satisfied

THE $u - v$ METHOD

- For a *basic feasible solution*, we find the dual variable u_i and v_j using the *complementary slackness conditions*

$$u_i + v_j = c_{ij} \qquad \forall \ basic \ x_{ij}$$

with u_i and v_j being unrestricted in sign

THE $u - v$ METHOD

- ❑ **We compute**

$$\tilde{c}_{ij} = c_{ij} - (u_i + v_j) \qquad\qquad \forall \ nonbasic \ x_{ij}$$

- ❑ **This step is the analogue of computing $\underline{\tilde{c}}^T$ in the simplex tableau approach (relative cost reduction vector)**

- ❑ **The *complementary-slackness*-based optimality test is performed :**

  $if \quad \tilde{c}_{ij} \geq 0 \quad \forall nonbasic \ x_{ij} \left[x_{ij} = 0 \right]$, **then the *basic feasible solution* is *optimal***

THE $u - v$ METHOD

- Otherwise, we consider all nonbasic variables $x_{\bar{p}\bar{q}}$ that satisfy

$$\tilde{c}_{\bar{p}\bar{q}} = c_{\bar{p}\bar{q}} - (u_{\bar{p}} + v_{\bar{q}}) < 0$$

and determine

$$\tilde{c}_{pq} = \underbrace{\min}_{\substack{\bar{p}\,\bar{q} \ \ni \ x_{\bar{p}\bar{q}} \\ \textit{is nonbasic} \\ \textit{and } \ \tilde{c}_{\bar{p}\bar{q}} < 0}} \left\{ \tilde{c}_{\bar{p}\,\bar{q}} \right\}$$

- We, then, select x_{pq} to become the next *basic variable* and repeat the process for this new *basic feasible solution and* continue the process until the *optimality conditions* are met

STP NUMERICAL EXAMPLE

- **We apply the $u - v$ scheme to the example previously discussed**

- **The basic step from the dual formulation is to require**

$$(u_i + v_j) = c_{ij} \qquad \forall \ nonbasic \ x_{ij}$$

STP NUMERICAL EXAMPLE

- ❑ We start with the *basic feasible solution* and apply the *complementary slackness conditions*

$$
\begin{array}{ccccccccc}
u_1 & + & v_4 & = & 1 & = & c_{14} \\
u_2 & + & v_4 & = & 4 & = & c_{24} \\
u_2 & + & v_3 & = & 5 & = & c_{23} \\
u_3 & + & v_2 & = & 6 & = & c_{32} \\
u_3 & + & v_1 & = & 7 & = & c_{31} \\
u_2 & + & v_1 & = & 10 & = & c_{21}
\end{array}
$$

- ❑ We have 6 equations in 7 unknowns and so there is an infinite number of solutions

STP NUMERICAL EXAMPLE

- ☐ **Arbitrarily, we set**

$$v_4 = 0$$

and solve the equations above to obtain

$$u_1 = 1$$

$$u_2 = 4$$

$$v_3 = 1$$

$$v_1 = 6$$

$$u_3 = 1$$

$$v_2 = 5$$

STP NUMERICAL EXAMPLE

- ☐ **The $\tilde{c}_{ij}$ for the *nonbasic variables* are**

$$x_{11}: \quad \tilde{c}_{11} = c_{11} - (u_1 + v_1) = 2 - (1+6) = -5$$

$$x_{12}: \quad \tilde{c}_{12} = c_{12} - (u_1 + v_2) = 2 - (1+5) = -4$$

$$x_{13}: \quad \tilde{c}_{13} = c_{13} - (u_1 + v_3) = 2 - (1+1) = 0$$

$$x_{34}: \quad \tilde{c}_{34} = c_{34} - (u_3 + v_4) = 8 - (1+0) = 7$$

$$x_{33}: \quad \tilde{c}_{33} = c_{33} - (u_3 + v_3) = 6 - (1+1) = 4$$

STP NUMERICAL EXAMPLE

- ☐ **We determine**

$$\tilde{c}_{pq} = \min_{\substack{\overline{p}\overline{q} \ni x_{\overline{p}\overline{q}} \\ \text{is nonbasic}}} = \tilde{c}_{11} = -5$$

and consequently we pick the *nonbasic variable* x_{11}

to enter the *basis*

- ☐ **We determine the maximal value of** x_{11} **and set**

$x_{11} = \theta$ **and make use of the tableau**

STP NUMERICAL EXAMPLE

market *j* / w/h *i*	M_1	M_2	M_3	M_4	a_i
W_1	θ			$3 - \theta$	3
W_2	$2 - \theta$		4	$1 + \theta$	7
W_3	2	3			5
b_j	4	3	4	4	

STP NUMERICAL EXAMPLE

- Therefore,

$$\theta = min\{ 2, 3 \} = 2$$

- Consequently, x_{21} becomes *0* and leaves the basis
- We obtain the *basic feasible solution*

$$x_{14} = 1, \ x_{11} = 2, \ x_{31} = 2, \ x_{32} = 3, \ x_{23} = 4, \ x_{24} = 3$$

and repeat to solve the $u - v$ problem for this new *basic feasible solution*

STP NUMERICAL EXAMPLE

market j *w/h i*	$v_1 = 2$	$v_2 = 1$	$v_3 = 2$	$v_4 = 1$	a_i
$u_1 = 0$	② 2	2	2	① 1	3
$u_2 = 3$	10	8	④ 5	③ 4	7
$u_3 = 5$	② 7	③ 6	6	8	5
b_j	4	3	4	4	

STP NUMERICAL EXAMPLE

- **The complementary slackness conditions of the nonzero valued basic variables obtain**

$$u_1 + v_1 = c_{11} = 2$$

$$u_1 + v_4 = c_{14} = 1$$

$$u_2 + v_3 = c_{23} = 5$$

$$u_2 + v_4 = c_{24} = 4$$

$$u_3 + v_1 = c_{31} = 7$$

$$u_3 + v_2 = c_{32} = 6$$

STP NUMERICAL EXAMPLE

- ☐ **We set**

$$u_1 = 0$$

and therefore

$$v_3 = 2 \qquad v_1 = 2$$

$$u_3 = 5 \qquad u_3 = 5$$

$$v_2 = 1 \qquad v_2 = 0$$

- ☐ **We compute $\tilde{c}_{ij}$ for each nonbasic variable x_{ij}**

STP NUMERICAL EXAMPLE

$$
\begin{aligned}
\tilde{c}_{12} &= c_{12} - (u_1 + v_2) = 2 - (0 + 1) = 1 \\
\tilde{c}_{13} &= c_{13} - (u_1 + v_3) = 2 - (0 + 2) = 0 \\
\tilde{c}_{21} &= c_{21} - (u_2 + v_1) = 10 - (3 + 2) = 5 \\
\tilde{c}_{22} &= c_{22} - (u_2 + v_2) = 8 - (3 + 1) = 4 \\
\tilde{c}_{33} &= c_{33} - (u_3 + v_3) = 6 - (5 + 2) = -1 \\
\tilde{c}_{34} &= c_{34} - (u_3 + v_4) = 8 - (5 + 1) = 2
\end{aligned}
$$

only possible improvement

- ☐ **We introduce x_{33} as a *basic variable* and determine its *nonnegative value* θ from the tableau**

STP NUMERICAL EXAMPLE

market j / *w/h i*	M_1	M_2	M_3	M_4	a_i
W_1	$2+\theta$			$1-\theta$	3
W_2			$4-\theta$	$3+\theta$	7
W_3	$2-\theta$	3	θ		5
b_j	4	3	4	4	

STP NUMERICAL EXAMPLE

- ☐ **The limiting value of θ is**

$$\theta = min \{ 2, 4, 1 \} = 1$$

- ☐ **Consequently, x_{14} leaves the basis and x_{33} enters the basis with the value 1**

- ☐ **We obtain the adjacent basic feasible solution in**

STP NUMERICAL EXAMPLE

market j w/h i	$v_1 = 2$	$v_2 = 1$	$v_3 = 1$	$v_4 = 0$	a_i
$u_1 = 0$	③ 2	2	2	1	3
$u_2 = 4$	10	8	③ 5	④ 4	7
$u_3 = 5$	① 7	③ 6	① 6	8	5
b_j	4	3	4	4	

STP NUMERICAL EXAMPLE

- **We evaluate $\tilde{c}_{ij}$ for each nonbasic variable; $\tilde{c}_{ij} \geq 0$ and so we have *an optimal solution* with**

  shipping 3 *from* W_1 *to* M_1 *with costs* 6

  shipping 1 *from* W_3 *to* M_1 *with costs* 7

  shipping 3 *from* W_3 *to* M_2 *with costs* 18

  shipping 1 *from* W_3 *to* M_3 *with costs* 6

  shipping 3 *from* W_2 *to* M_3 *with costs* 15

  shipping 4 *from* W_2 *to* M_4 *with costs* 16

  and resulting in the *least total costs* of 68

ELECTRICITY DISTRIBUTION EXAMPLE

- ❑ **We consider an electric utility system in which 3 power plants are used to supply the electricity demand of 4 cities**
- ❑ **The supplies available from the 3 plants are given**
- ❑ **The demands of the 4 cities are specified**
- ❑ **The costs of supply per 10^6 kWh are given**

ELECTRICITY COSTS

to *from*		*city*				*supplies* (10^6 *kWh*)
		1	2	3	4	
plant	1	8	6	10	9	35
	2	9	12	13	7	50
	3	14	9	16	5	40
demands (10^6 *kWh*)		45	20	30	30	125

ELECTRICITY COSTS

balanced transportation problem

to / *from*	*city* 1	2	3	4	*supplies* (10^6 *kWh*)
			0	9	35
			3	7	50
	14	9	16	5	40
demands (10^6 *kWh*)	45	20	30	30	125

ELECTRICITY ALLOCATION EXAMPLE

- **We note that**

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

and so we have a balanced transportation problem

- **We find a *basic feasible solution* using the least-cost rule**

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to *from*		*city*				*supplies* ($10^6 kWh$)
		1	2	3	4	
plant	1	8	6	10	0 9	35
	2	9	12	13	0 7	50
	3	14	9	16	30 5	10
demands ($10^6 kWh$)		45	20	30	30	125

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

- ☐ And we set

$$x_{34} = 30$$

$$x_{14} = 0$$

$$x_{24} = 0$$

- ☐ We compute the remaining supply at plant 3 and remove column corresponding to city 4 from further consideration
- ☐ We continue with the reduced system

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to / *from*		*city*			*supplies* $(10^6\ kWh)$
		1	2	3	
plant	1	8	20 6	10	15
	2	9	0 12	13	50
	3	14	0 9	16	10
demands $(10^6\ kWh)$		45	20	30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and so we set

$$x_{12} = 20$$

$$x_{22} = 0$$

$$x_{32} = 0$$

- ☐ We recompute the supply remaining at plant 1 and remove column corresponding to city 2
- ☐ The new reduced system obtains

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to *from*		*city*		*supplies* (10^6 *kWh*)
		1	3	
plant	1	15 8	0 10	15
	2	9	13	50
	3	14	16	10
demands (10^6 *kWh*)		30	30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and therefore we set

$$x_{11} = 15$$

$$x_{13} = 0$$

and remove the row corresponding to plant 1 from further consideration since its supply is exhausted

- ☐ The operation is repeated on the reduced system

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to from		city		supplies $(10^6 \, kWh)$
		1	3	
plant	2	30 9	13	20
	3	0 14	16	10
demands $(10^6 \, kWh)$		30	30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and therefore we set

$$x_{21} = 30$$

$$x_{31} = 0$$

and remove the column corresponding to city 1

from further consideration

- ☐ We are finally left with

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to from		city	supplies (10^6 kWh)
		3	
plant	2	20 13	20
	3	10 16	10
demands (10^6 kWh)		30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

which allows us to set

$$x_{23} = 20$$

$$x_{33} = 10$$

- ☐ The basic feasible solution has the costs

$$Z = 30 \cdot 5 + 20 \cdot 6 + 15 \cdot 8 + 30 \cdot 9 + 20 \cdot 13 + 10 \cdot 16 = 1{,}080$$

- ☐ We improve this solution by using the $u - v$ *scheme*
- ☐ The first tableau corresponding to the initial basic feasible solution is:

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to *from*		*city*				*supplies* $(10^6\,kWh)$
		1	2	3	4	
plant	1	⑮ 8	⑳ 6			35
	2	㉚ 9		⑳ 13		50
	3			⑩ 16	㉚ 5	40
demands $(10^6\,kWh)$		45	20	30	30	

STP NUMERICAL EXAMPLE

- We compute, the possible improvements at each nonbasic variable:

$$\begin{array}{lllllllll}
\tilde{c}_{31} & = & c_{31} & - & (u_3 + v_1) & = & 14 & - & (4+8) & = & 2 \\
\tilde{c}_{22} & = & c_{22} & - & (u_2 + v_2) & = & 12 & - & (1+6) & = & 5 \\
\tilde{c}_{32} & = & c_{32} & - & (u_3 + v_2) & = & 9 & - & (4+6) & = & -1 \\
\tilde{c}_{13} & = & c_{13} & - & (u_1 + v_3) & = & 10 & - & (0+12) & = & -2 \\
\tilde{c}_{14} & = & c_{14} & - & (u_1 + v_4) & = & 9 & - & (0+1) & = & 8 \\
\tilde{c}_{24} & = & c_{24} & - & (u_2 + v_4) & = & 7 & - & (1+1) & = & 5
\end{array}$$

improvement possible

better improvement

STP NUMERICAL EXAMPLE

- ❑ **We bring x_{13} into the basis and determine the value of θ using the tableau structure**
- ❑ **From the tableau we conclude that**

$$\theta = \min \{ 15, 20 \} = 15$$

and therefore x_{11} leaves the basis to determine the *adjacent basic feasible solution* given in the table

STP NUMERICAL EXAMPLE

cities / *plants*	1	2	3	4	a_i
1	$15-\theta$	20	θ		35
2	$30+\theta$		$20-\theta$		50
3			10	30	40
b_j	45	20	30	30	

STP NUMERICAL EXAMPLE

- ☐ The adjacent basic feasible solution is

$$x_{21} = 45, \quad x_{12} = 20, \quad x_{13} = 15, \quad x_{23} = 5, \quad x_{33} = 10, \quad x_{34} = 30$$

and the new value of Z is

$$Z = 20 \cdot 6 + 15 \cdot 10 + 45 \cdot 9 + 5 \cdot 13 + 10 \cdot 16 + 30 \cdot 5$$

$$= 1050 < 1080$$

- ☐ We again pursue a $u - v$ improvement strategy by starting with the tableau

STP NUMERICAL EXAMPLE

cities *plants*	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = -1$	*supplies*
$u_1 = 0$		20 $\boxed{6}$	15 $\boxed{10}$		35
$u_2 = 3$	45 $\boxed{9}$		5 $\boxed{13}$		50
$u_3 = 6$			10 $\boxed{16}$	30 $\boxed{5}$	40
demands	45	20	30	30	

STANDARD TRANSPORTATION EXAMPLE

- The complementary slackness conditions obtain the possible improvements

$$
\begin{array}{lclclclclclcr}
\tilde{c}_{11} & = & c_{11} & - & (u_1+v_1) & = & 8 & - & (0+6) & = & 2 \\
\tilde{c}_{31} & = & c_{31} & - & (u_3+v_1) & = & 14 & - & (6+6) & = & 2 \\
\tilde{c}_{22} & = & c_{22} & - & (u_2+v_2) & = & 12 & - & (3+6) & = & 3 \\
\tilde{c}_{32} & = & c_{32} & - & (u_3+v_2) & = & 9 & - & (6+6) & = & \boxed{-3} \\
\tilde{c}_{14} & = & c_{14} & - & (u_1+v_4) & = & 9 & - & (0-1) & = & 10 \\
\tilde{c}_{24} & = & c_{24} & - & (u_2+v_4) & = & 7 & - & (3-1) & = & 5
\end{array}
$$

only possible improvement

- We bring x_{32} into the basis and with its value θ determined from

STP NUMERICAL EXAMPLE

plants / *cities*	1	2	3	4	a_i
1		$20 - \theta$	$15 + \theta$		35
2	45		5		50
3		θ	$10 - \theta$	30	40
b_j	45	20	30	30	

STP NUMERICAL EXAMPLE

and so

$$\theta = min\,\{\, 10, 20 \,\} = 10$$

- ☐ The adjacent basic feasible solution is, then,

$$x_{21} = 45 \qquad x_{12} = 10 \qquad x_{32} = 10$$

$$x_{13} = 25 \qquad x_{23} = 5 \qquad x_{34} = 30$$

and the value of Z becomes

$$Z = 45 \cdot 9 + 10 \cdot 6 + 10 \cdot 9 + 25 \cdot 10 + 5 \cdot 13 + 30 \cdot 5 = 1{,}020$$

- ☐ You are asked to prove, using complementary slackness conditions, that this is the optimum

NONSTANDARD TRANSPORTATION PROBLEM

- **The nonstandard transportation problem arises when supply and demand are unbalanced: either supply exceeds demand or vice versa**
- **We solve by transforming the nonstandard problem into a standard one**
- **The approach is to create a** ***fictitious*** **entity – a market to absorb the surplus supply or a warehouse for the supply deficit – and solve the problem with the fictitious entity as a balanced problem**

NONSTANDARD TRANSPORTATION PROBLEM

- **For the case**

$$\underbrace{\sum_{i=1}^{m} a_i}_{\textbf{supply}} > \underbrace{\sum_{i=1}^{n} b_j}_{\textbf{demand}}$$

we create the fictitious market M_{n+1} to absorb all the excess supply $\left(\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j \right)$; we set $c_{i,n+1} = 0$, $\forall \; i = 1, 2, \ldots, m$ since M_{n+1} is fictitious

- **The problem is then in standard form with $j = 1, 2, \ldots, n, n+1$, for the augmented number of markets**

NONSTANDARD TRANSPORTATION PROBLEM

- **For the case**

$$\underbrace{\sum_{j=1}^{n} b_j}_{\textbf{demand}} > \underbrace{\sum_{i=1}^{m} a_i}_{\textbf{supply}}$$

the problem is *not* , in effect, *feasible* since all the demands cannot be met and therefore the least-cost shipping schedule is that which will supply as much as possible of the demands of the markets at the lowest cost

NONSTANDARD TRANSPORTATION PROBLEM

- ☐ **For the excess demand case, we introduce the fictitious warehouse** W_{m+1} **to supply the shortage** $\left[\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \right]$ **and we set** $c_{m+1,j} = 0, \; j = 1, 2, \ldots , n$

- ☐ **The problem is in standard form with** $i = 1, \ldots ,$ $m + 1$ **(number of warehouses augmented by 1)**

NONSTANDARD TRANSPORTATION PROBLEM

- ☐ **Note that the variable $x_{m+1,\,j}$ is the *shortage* at market j and is the shortfall in the demand b_j experienced by each market M_j due to inadequacy of the supplies $j = 1, 2, \ldots, n$**
- ☐ **For each market j, $x_{m+1,\,j}$ provides the measure of the *infeasibility* of the problem**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **This problem is concerned with the scheduling the purchases of 2 plants – A and B – of the raw supplies from 3 growers with given availability / price**

grower	*availability* (*ton*)	*price* (*$ / ton*)
Smith	200	10
Jones	300	9
Richard	400	8

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **The shipping costs in** $\$/ton$ **are given by**

to / *from*	*plant*	
	A	***B***
Smith	**2**	**2.5**
Jones	**1**	**1.5**
Richard	**5**	**3**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **The plants’ capacity limits and labor costs are**

plant	*A*	*B*
capacity *(ton)*	**450**	**550**
labor costs *($ / ton)*	**25**	**20**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **The competitive selling price for canned goods is $50\ \$ / ton$ and the company can sell all it produces**
- ❑ **The problem is to determine the purchase schedule that produces the *maximum* profits**
- ❑ **Note that this is an unbalanced problem since**

$$\begin{aligned} supply &= 200 + 300 + 400 = 900\ tons \\ demand &= 450 + 550 = 1000\ tons > 900\ tons \end{aligned}$$

- ❑ **The decision variables are the amounts bought from each grower and shipped to each plant**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ❑ **The objective is formulated as**

$$\max Z = \left[\underbrace{50-25-10-2}_{13}\right]x_{SA} + \left[\underbrace{50-25-9-1}_{15}\right]x_{JA}$$

$$+ \left[\underbrace{50-25-8-5}_{12}\right]x_{RA} + \left[\underbrace{50-20-10-2.5}_{17.5}\right]x_{SB}$$

$$+ \left[\underbrace{50-20-9-1.5}_{19.5}\right]x_{JB} + \left[\underbrace{50-20-8-3}_{19}\right]x_{RB}$$

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ☐ **The supply constraints are**

$$x_{SA} + x_{SB} \leq 200$$

$$x_{JA} + x_{JB} \leq 300$$

$$x_{RA} + x_{RB} \leq 400$$

- ☐ **The demand constraints are**

$$x_{SA} + x_{JA} + x_{RA} \leq 450$$

$$x_{SB} + x_{JA} + x_{RB} \leq 550$$

EXAMPLE: CANNING OPERATIONS SCHEDULING

- ☐ **Clearly, all decision variables are nonnegative**
- ☐ **The unbalanced nature of the problem requires the introduction of a *fictitious* grower *F*, who is able to supply 100 *tons* of the supply shortage; the addition of *F* allows the *nonstandard* problem to be stated as a *standard transportation problem***
- ☐ **We set up the *STP* tableau**

EXAMPLE: CANNING OPERATIONS SCHEDULING

plant j / *grower i*	*A*	*B*	*supply*
S	13	17.5	200
J	15	19.5	300
R	12	19	400
F	*0*	*0*	100
demand	450	550	1,000

EXAMPLE: CANNING OPERATIONS SCHEDULING

- **In this problem, the objective is a *maximization* rather than a *minimization***
- **We therefore recast the "mechanics" of the $u - v$ scheme for the *maximization* problem**
- **As a homework exercise, show that the duality complementary slackness conditions allow us to change the $u - v$ algorithm in the following way:**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- the selection of the nonbasic variable x_{ij} to enter the basis is from those x_{ij} whose corresponding

$$c_{ij} > u_i + v_j$$

  and we focus on and evaluate all $\tilde{c}_{ij} > 0$ for which x_{ij} is a candidate to enter the basis

- we pick x_{pq} corresponding to

$$\tilde{c}_{pq} = \underbrace{\max}_{\substack{\bar{p}\bar{q} \ni x_{\bar{p}\bar{q}} \\ \text{is nonbasic} \\ \text{and } \tilde{c}_{\bar{p}\bar{q}} > 0}} \left\{ \tilde{c}_{\bar{p}\bar{q}} \right\}$$

EXAMPLE SOLUTION

plant j *grower i*	*A*	*B*	*supply*
S	200 13	*0* 17.5	200
J	250 15	50 19.5	300
R	*0* 12	400 19	400
F	*0* *0*	100 *0*	100
demand	450	550	

EXAMPLE SOLUTION

- ☐ We construct the $u - v$ relations for this solution

$$u_1 + v_1 = 13 \qquad u_2 + v_2 = 19.5$$

$$u_2 + v_1 = 15 \qquad u_3 + v_2 = 19$$

$$u_4 + v_2 = 0$$

- ☐ We arbitrarily set $u_1 = 0$ and compute

$$v_1 = 13,\ u_2 = 2,\ v_2 = 17.5,\ u_3 = 1.5,\ u_4 = -17.5$$

EXAMPLE SOLUTION

- We evaluate the $\tilde{c}_{ij}$ corresponding to the nonbasic variables

$$\tilde{c}_{31} = c_{31} - (u_3 + v_1) = 12 \; - \; (1.5 + 13) \; = - \; 2.5$$

$$\tilde{c}_{41} = c_{41} - (u_4 + v_1) = \mathit{0} \; - (-17.5 + 13) \; = \; 4.5$$

$$\tilde{c}_{12} = c_{12} - (u_1 + v_2) = 17.5 - \; (\mathit{0} + 17.5) \; = \; \mathit{0}$$

single possible improvement

- Thus, x_{41} enters the basis and we determine θ

EXAMPLE SOLUTION

plant j *grower i*	*A*	*B*	*supply*
S	200 13		200
J	$250 - \theta$ 15	$50 + \theta$ 19.5	300
R		400 19	400
F	θ *0*	$100 - \theta$ *0*	100
demand	450	550	

EXAMPLE SOLUTION

- ☐ It follows that

$$\theta = min \; \{ 250, 100 \} = 100$$

and so the adjacent basic feasible solution is

$$x_{11} = 200, \; x_{21} = 150, \; x_{41} = 100, \; x_{22} = 150, \; x_{32} = 400$$

- ☐ We repeat the $u - v$ procedure with the new *basic variables* and solve

EXAMPLE SOLUTION

$u_1 + v_1 = 13$

$u_2 + v_2 = 19.5$

$u_2 + v_1 = 15$

$u_3 + v_2 = 19$

$u_4 + v_1 = 0$

- ☐ **We solve by arbitrarily setting $u_1 = 0$ and obtain**

$$v_1 = 13 \, , \; u_2 = 2 \, , \; v_2 = 17.5 \, , \; u_3 = 1.5 \, , \; u_4 = -13$$

EXAMPLE SOLUTION

- ☐ **We compute the $\tilde{c}_{ij}$ for the nonbasic variables**

$$\tilde{c}_{12} = 17.5 - (0 + 17.5) = 0$$

$$\tilde{c}_{31} = 12 - (1.5 + 13) = -2.5$$

$$\tilde{c}_{42} = 0 - (-13 + 17.5) = -4.5$$

EXAMPLE SOLUTION

- ☐ Since each $\tilde{c}_{ij}$ is ≤ 0 , no improvement in the maximization is possible and so the maximum profits are

$$Z = (200)13 + (150)15 + (100)0 + (150)19.5 + (400)19$$

$$= 15,375\ \$$$

SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

- ☐ **The problem is concerned with the weekly production scheduling over a 4 – week period**
  - ○ **production costs for each item**

first two weeks	**$ 10**
last two weeks	**$ 15**

  - ○ **demands that need to be met are**

week	**1**	**2**	**3**	**4**
demand	**300**	**700**	**900**	**800**

SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

- ❍ weekly plant capacity is 700
- ❍ overtime is possible for weeks 2 and 3 with
  - – the production of additional 200 *units*
  - – additional cost per unit of *$* 5
- ❍ *$* 3 for weekly storage of unsold production
- ❍ the objective is to *minimize* the *total costs* for the 4-week schedule

❑ The decision variables are

x_{ij} = *production in week i for use in week j market*

SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

demand wk. / *production wk.*		1	2	3	4	*F*	*supply*
1		10	13	16	19	*0*	700
2	*normal*	*M*	10	13	16	*0*	700
	o/t	*M*	15	18	[illegible]	*0*	200
3	*normal*	*M*	*M*	15	18	*0*	700
	o/t	[illegible]	[illegible]	[illegible]	[illegible]	*0*	200
4		*M*	*M*	*M*	15	*0*	700
demand		300	700	900	800	500	

M is a very large number

3,200

2,700

3,200 – 2,700

ASSIGNMENT PROBLEM

- We are given

n machines $M_1, M_2, \ldots, M_n \leftrightarrow i$

n jobs $J_1, J_2, \ldots, J_n \leftrightarrow j$

$c_{ij} =$ cost of doing job j on machine i

$c_{ij} = M$ if job j cannot be done on machine i

each machine can only do one job and we wish to determine the optimal match, i.e., the assignment with the lowest total costs of doing each job j on the n available machines

ASSIGNMENT PROBLEM

- The brute force approach is simply enumeration: consider $n = 10$ and there are $3,628,800$ possible choices!

- We can, however, introduce *categorical* decision variables

$$x_{ij} = \begin{cases} 1 & \textit{job } j \textit{ is assigned to machine } i \\ 0 & \textit{otherwise} \end{cases}$$

ASSIGNMENT PROBLEM

and the problem constraints can be stated as

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \text{ each machine does exactly } 1 \text{ job}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \text{ each job is assigned to } 1 \text{ machine}$$

- ☐ **The objective, then, is**

$$min\ Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

ASSIGNMENT PROBLEM

- ☐ **This assignment problem is an *STP* with**

$$a_i = 1 \qquad \forall\, i$$

$$b_j = 1 \qquad \forall\, j$$

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j$$

NONSTANDARD ASSIGNMENT PROBLEM

- **Suppose we have m machines and n jobs with $m \neq n$**
- **We may convert this into an equivalent *standard assignment problem* with *equal* number of machines and jobs**
- **The conversion requires the introduction of either *fictitious* jobs or *fictitious* machines**

NONSTANDARD ASSIGNMENT PROBLEM

- **In the case $m > n$:**

  we create ($m - n$) fictitious jobs and we have m machines and $n + m - n = m$ jobs; we assign the machinery costs for the fictitious goods to be 0 : note that there is no change in the objective function since a fictitious job assigned to a machine is, in effect, a machine that remains *idle*

NONSTANDARD ASSIGNMENT PROBLEM

- ☐ **For the case $n > m$:**

  we create ($n - m$) *fictitious* machines with machine costs of 0 and the solution obtained has the ($n - m$) jobs that cannot be done due to lack of machines

NONSTANDARD ASSIGNMENT PROBLEM

- ☐ **In principle, any assignment problem may be solved using the transportation problem technique; in practice, this approach is not practical since there exists *degeneracy* in every basic feasible solution**
- ☐ **We note that in the *standard assignment problem* for m machines with $m = n$, there are exactly m x_{ij} that are 1 (*nonzero*) but *every basic feasible solution* of the transportation problem has $(2m - 1)$ basic variables of which $(m - 1)$ have the value zero**